Monday, September 28, 2015

Page 452

Problem 1

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the x-axis.

Solution. The radius is $R(x) = -x + 1$, so the volume is

$$\begin{aligned} V &= \int_0^1 \pi(-x+1)^2 \, dx \\ &= \pi \int_0^1 (x^2 - 2x + 1) \, dx \\ &= \pi \left[\frac{1}{3}x^3 - x^2 + x \right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{2} + 1 \right) \\ &= \frac{5\pi}{6}. \end{aligned}$$

Problem 2

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the x-axis.

Solution. The radius is $R(x) = 4 - x^2$, so the volume is

$$\begin{aligned} V &= \int_0^2 \pi(4 - x^2)^2 \, dx \\ &= \pi \int_0^2 (16 - 8x^2 + x^4) \, dx \\ &= \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 \\ &= \pi \left(32 - \frac{64}{3} + \frac{32}{5} \right) \\ &= \frac{256\pi}{15}. \end{aligned}$$

Problem 3

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the x-axis.

Solution. The radius is $R(x) = \sqrt{x}$, so the volume is

$$\begin{aligned} V &= \int_1^4 \pi(\sqrt{x})^2 \, dx \\ &= \pi \int_1^4 x \, dx \\ &= \pi \left[\frac{1}{2}x^2\right]_1^4 \\ &= \pi \left(8 - \frac{1}{2}\right) \\ &= \frac{15\pi}{2}. \end{aligned}$$

Problem 7

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the y-axis.

Solution. The function is $y = x^2$. The radius is measured horizontally, so we must solve for x: $x = \sqrt{y}$. Then the radius is $R(y) = \sqrt{y}$. The volume is

$$\begin{aligned} V &= \int_0^4 \pi(\sqrt{y})^2 \, dy \\ &= \pi \int_0^4 y \, dy \\ &= \pi \left[\frac{1}{2}y^2\right]_0^4 \\ &= 8\pi. \end{aligned}$$

Problem 8

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region about the y-axis.

Solution. The function is $y = \sqrt{16 - x^2}$. The radius is measured horizontally, so we must solve for x: $x = \sqrt{16 - y^2}$. Then the radius is $R(y) = \sqrt{16 - y^2}$. The volume is

$$\begin{aligned} V &= \int_0^4 \pi(\sqrt{16 - y^2})^2 \, dy \\ &= \pi \int_0^4 (16 - y^2) \, dy \\ &= \pi \left[16y - \frac{1}{3}y^3 \right]_0^4 \\ &= \pi \left(64 - \frac{64}{3} \right) \\ &= \frac{128\pi}{3}. \end{aligned}$$

Problem 11(a)

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$\begin{aligned} y &= \sqrt{x}, \\ y &= 0, \\ x &= 3 \end{aligned}$$

about the x-axis.

Solution. The radius is measured vertically (perpendicular to the x-axis), so it is $R(x) = \sqrt{x}$. The volume is

$$\begin{aligned} V &= \int_0^3 \pi(\sqrt{x})^2 \, dx \\ &= \pi \int_0^3 x \, dx \\ &= \pi \left[\frac{1}{2}x^2 \right]_0^3 \\ &= \frac{9\pi}{2}. \end{aligned}$$

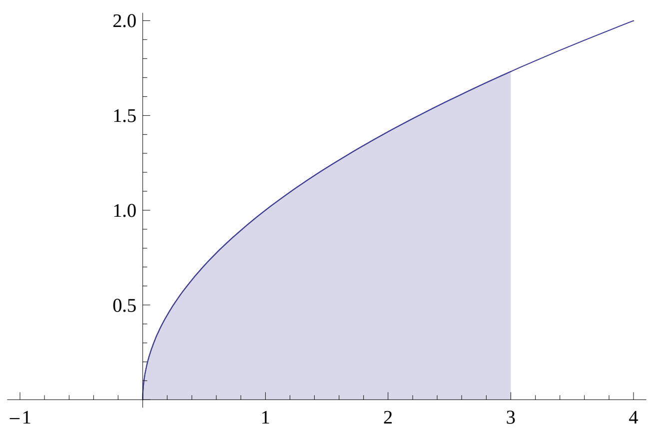
Problem 11(c)

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$\begin{aligned} y &= \sqrt{x}, \\ y &= 0, \\ x &= 3 \end{aligned}$$

about the line $x = 3$.

Solution. The radius is measured horizontally (perpendicular to the line $x = 3$), so it is $R(y) = 3 - y^2$. The extremities in the y direction are 0 and $\sqrt{3}$.



The volume is

$$\begin{aligned} V &= \int_0^{\sqrt{3}} \pi(3 - y^2)^2 \, dy \\ &= \pi \int_0^{\sqrt{3}} (9 - 6y^2 + y^4) \, dy \\ &= \pi \left[9y - 2y^3 + \frac{1}{5}y^5 \right]_0^{\sqrt{3}} \\ &= \pi \left(9\sqrt{3} - 6\sqrt{3} + \frac{9}{5}\sqrt{3} \right) \\ &= \frac{24\sqrt{3}\pi}{5}. \end{aligned}$$

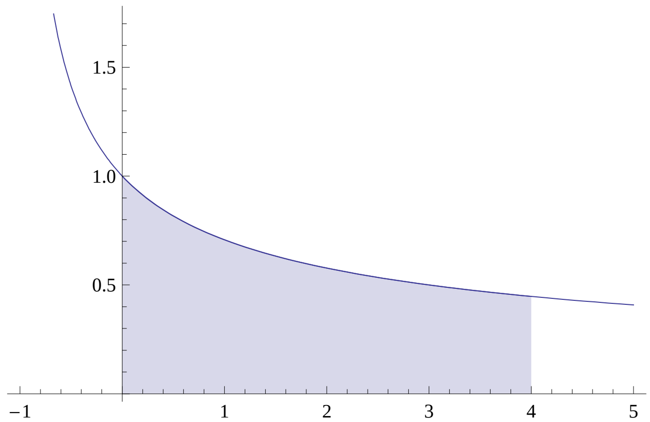
Problem 23

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$y = \frac{1}{\sqrt{x+1}},$$
$$y = 0,$$
$$x = 0,$$
$$x = 4$$

about the x-axis.

Solution. The graph is



The radius is $R(x) = \dfrac{1}{\sqrt{x+1}}$. The volume is

$$\begin{aligned} V &= \int_0^4 \pi \left(\frac{1}{\sqrt{x+1}} \right)^2 \, dx \\ &= \pi \int_0^4 \frac{1}{x+1} \, dx \\ &= \pi \left[\ln |x+1| \right]_0^4 \\ &= \pi \left(\ln 5 - \ln 1 \right) \\ &= \pi \ln 5. \end{aligned}$$

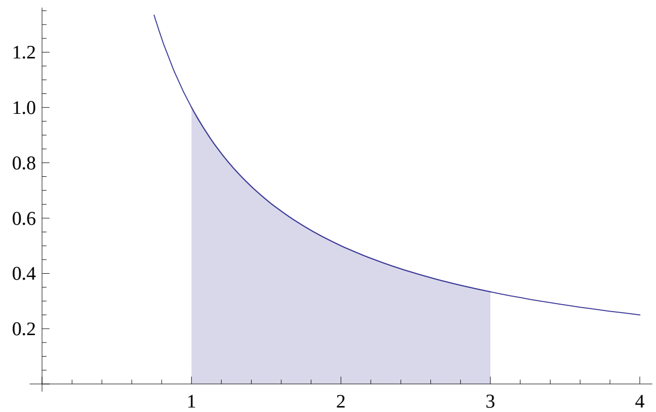
Problem 25

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$y = \frac{1}{x},$$
$$y = 0,$$
$$x = 1,$$
$$x = 3$$

about the x-axis.

Solution. The graph is



The radius is $R(x) = \frac{1}{x}$. The volume is

$$\begin{aligned} V &= \int_1^3 \pi \left(\frac{1}{x}\right)^2 \, dx \\ &= \pi \int_1^3 \frac{1}{x^2} \, dx \\ &= \pi \left[-\frac{1}{x}\right]_1^3 \\ &= \pi \left(-\frac{1}{3} - (-1)\right) \\ &= \frac{2\pi}{3}. \end{aligned}$$

Problem 33

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$
\begin{aligned}
y &= \sin x, \\
y &= 0, \\
x &= 0, \\
x &= \pi
\end{aligned}
$$

about the x-axis.

Solution. The radius is $R(x) = \sin x$. The volume is

$$
\begin{aligned}
V &= \int_0^\pi \pi \left(\sin x\right)^2 \ dx \\
&= \pi \int_0^\pi \sin^2 x \ dx
\end{aligned}
$$

Oops. We have not yet learned how to integrate $\sin^2 x$. The key is to use the identity

$$\cos 2x = 1 - 2\sin^2 x.$$

Solve for $\sin^2 x$ to get

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Now we can find the volume.

$$
\begin{aligned}
V &= \int_0^\pi \frac{1}{2}(1 - \cos 2x) \ dx \\
&= \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) \ dx \\
&= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi \\
&= \frac{\pi}{2} \left(\left(\pi - \frac{1}{2} \sin 2\pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right) \\
&= \frac{\pi^2}{2}.
\end{aligned}
$$

Problem 35

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$y = e^{x-1},$$
$$y = 0,$$
$$x = 1,$$
$$x = 2$$

about the x-axis.

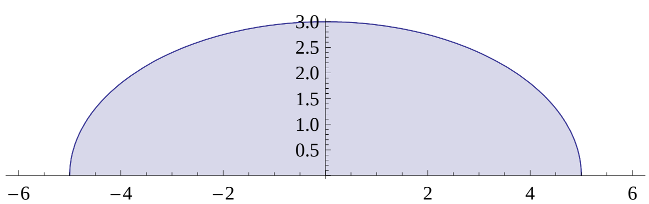
Solution. The radius is $R(x) = e^{x-1}$. The volume is

$$\begin{aligned} V &= \int_0^2 \pi \left(e^{x-1}\right)^2 \, dx \\ &= \pi \int_0^2 e^{2x-2} \, dx \\ &= \pi \left[\frac{1}{2} e^{2x-2}\right]_0^2 \\ &= \pi \left(\frac{1}{2} e^2 - \frac{1}{2} e^{-2}\right) \\ &= \frac{\pi}{2}(e^2 - e^{-2}). \end{aligned}$$

Problem 65

Problem. Find the volumes of the solids generated if the upper half of the ellipse $9x^2 + 25y^2 = 225$ is revolved about (a) the x-axis to form a prolate spheroid, and (b) the y-axis to form an oblate spheroid.

Solution. The graph is



(a) If we rotate about the x-axis, then the radius is

$$R(y) = 3\sqrt{1 - \frac{x^2}{25}}$$

and the volume is

$$\begin{aligned}
V &= \int_{-5}^{5} \pi \left(3\sqrt{1-\frac{x^2}{25}}\right)^2 \, dx \\
&= 9\pi \int_{-5}^{5} \left(1-\frac{x^2}{25}\right) \, dx \\
&= 9\pi \left[y - \frac{1}{75}x^3\right]_{-5}^{5} \\
&= 9\pi \left(\left(5-\frac{125}{75}\right) - \left((-5)+\frac{125}{75}\right)\right) \\
&= 60\pi.
\end{aligned}$$

(b) If we rotate about the y-axis, then the radius is

$$R(x) = 5\sqrt{1-\frac{y^2}{9}}$$

and the volume is

$$\begin{aligned}
V &= \int_{-3}^{3} \pi \left(5\sqrt{1-\frac{y^2}{9}}\right)^2 \, dy \\
&= 25\pi \int_{-3}^{3} \left(1-\frac{y^2}{9}\right) \, dy \\
&= 25\pi \left[x - \frac{1}{27}y^3\right]_{-3}^{3} \\
&= 25\pi \left((3-1) - ((-3)+1)\right) \\
&= 100\pi.
\end{aligned}$$